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Normal ordering in the theory of correlation functions of exactly solvable models

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Abstract. We study models of quantum-statistical mechanics which can be solved by the algebraic Bethe ansatz. The general method of calculating correlation functions is based on the method of determinant representations. The auxiliary Fock space and auxiliary Bose fields are introduced in order to remove the two-body scattering and represent correlation functions as a mean value of a determinant of a Fredholm integral operator; the representation has a simple form for large space and time separations. In this paper we explain how to calculate the mean value in the auxiliary Fock space of an asymptotic expression of the Fredholm determinant. It is necessary for the evaluation of the asymptotic form of the physical correlation functions.

1. Introduction

In this paper we use an example of the quantum nonlinear Schrödinger equation, i.e. the one-dimensional Bose gas with delta-function interactions, in order to illustrate the development in the theory of quantum correlation functions. The correlation function of local fields in this model was studied in [1–4], and its determinant representation was obtained in [1]. The representation of the correlation function in terms of the Fredholm determinant of a linear integral operator is the basis of our approach. The differential equations for the Fredholm determinant were obtained in [2]. They are directly related to the classical nonlinear Schrödinger equation. These differential equations were solved in the asymptotic regime of large space and time separations; the simplified asymptotic form of the Fredholm determinant was obtained in [3, 4]. This expression is an operator in an auxiliary Fock space. In order to find the asymptotics of the correlation function, one should calculate the vacuum mean value of this expression. It is a necessary step in the calculation of asymptotics of physical correlation functions. The problem of evaluating the vacuum expectation (mean value) is a combinatorial problem, closely related to the procedure of the normal ordering in the quantum field theory. In this paper we study just this problem.

We briefly recall the basic definitions of the model under consideration for the reader's convenience. The quantum nonlinear Schrödinger equation can be described in terms of the canonical Bose fields $\psi(x,t)$ and $\psi^\dagger(x,t)$ ($x \in R$) obeying the equal time commutation relations

$$[\psi(x,t), \psi^\dagger(y,t)] = \delta(x-y). \tag{1.1}$$

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The Hamiltonian and momentum of the model are

$$H = \int \mathrm{d}x\ (\partial_x \psi^\dagger(x) \partial_x \psi(x) + c\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x) - h\psi^\dagger(x)\psi(x)) \tag{1.2}$$

and

$$P = -\mathrm{i} \int \mathrm{d}x\ \psi^\dagger(x)\partial_x \psi(x). \tag{1.3}$$

Here $0 < c \leqslant \infty$ is the coupling constant and h is the chemical potential. The spectrum of the model was first described by Lieb and Liniger [5] and Lieb [6]. The Lax representation for the corresponding classical equation of motion

$$\mathrm{i}\frac{\partial}{\partial t}\psi = [\psi, H] = -\frac{\partial^2}{\partial x^2}\psi + 2c\psi^\dagger\psi\psi - h\psi \tag{1.4}$$

was found by Zakharov and Shabat [7]. The quantum inverse scattering method for the model was formulated by Faddeev and Sklyanin [8].

The quantum nonlinear Schrödinger equation is equivalent to the Bose gas with delta-function interactions. In the sector with N particles the Hamiltonian of the Bose gas is given by

$$\mathcal{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial z_j^2} + 2c \sum_{1 \leqslant j < k \leqslant N} \delta(z_j - z_k) - Nh. \tag{1.5}$$

In this paper we shall consider the thermodynamics of this model. The partition function and free energy of the model are defined by

$$Z = \mathrm{tr}\, \mathrm{e}^{-\frac{H}{T}} = \mathrm{e}^{-\frac{F}{T}}. \tag{1.6}$$

The free energy F has been explicitly represented in terms of the Yang–Yang [9] equation

$$\varepsilon(\lambda) = \lambda^2 - h - \frac{T}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\mu\, \frac{2c}{c^2 + (\lambda - \mu)^2} \ln(1 + \mathrm{e}^{-\frac{\varepsilon(\mu)}{T}}) \tag{1.7}$$

$$F = -\frac{T}{2\pi} \int_{-\infty}^{\infty} \ln(1 + \mathrm{e}^{-\frac{\varepsilon(\mu)}{T}}). \tag{1.8}$$

The correlation function studied in this paper is defined by

$$\langle \psi(0,0)\psi^\dagger(x,t) \rangle_T = \frac{\mathrm{tr}(\mathrm{e}^{-\frac{H}{T}} \psi(0,0)\psi^\dagger(x,t))}{\mathrm{tr}(\mathrm{e}^{-\frac{H}{T}})}. \tag{1.9}$$

This paper is organized as follows. In section 2 we shall remind the reader of the asymptotic expression for the Fredholm determinant that represents the correlation function. We shall also describe its dependence on the quantum fields. Section 3 is devoted to the main results of the paper. In it we develop a technique for the evaluation of the mean values in auxiliary Fock space. It is related to the problems of the normal ordering in quantum field theory. In section 4 we use this technique to calculate the mean value of the asymptotic expressions for the Fredholm determinant.

2. Asymptotics of the Fredholm determinant

In order to find the determinant representation of the correlation functions one has to introduce an auxiliary Fock space and three Bose fields $\psi(\lambda)$, $\phi(\lambda)$, and $\Phi(\lambda)$, which are linear combinations of annihilation and creation operators $p(\lambda)$ and $q(\lambda)$:

$$\begin{aligned}\psi(\lambda) &= q_\psi(\lambda) + p_\psi(\lambda)\\ \phi(\lambda) &= q_\phi(\lambda) + p_\phi(\lambda)\\ \Phi(\lambda) &= q_\Phi(\lambda) + p_\Phi(\lambda).\end{aligned} \tag{2.1}$$

The operators $p(\lambda)$ annihilate the Fock vacuum

$$p(\lambda)|0) = 0 \tag{2.2}$$

and the corresponding creation operators $q(\lambda)$ annihilate the dual vacuum

$$(0|q(\lambda) = 0. \tag{2.3}$$

We shall also use the function

$$h(\lambda, \mu) = (\lambda - \mu + \mathrm{i}c)/\mathrm{i}c \tag{2.4}$$

which enters into the non-vanishing commutators

$$[p_\psi(\lambda), q_\phi(\mu)] = -[p_\phi(\lambda), q_\psi(\mu)] = \ln\left(\frac{h(\mu,\lambda)}{h(\lambda,\mu)}\right) \tag{2.5}$$

$$[p_\psi(\lambda), q_\Phi(\mu)] = [p_\Phi(\lambda), q_\psi(\mu)] = [p_\psi(\lambda), q_\psi(\mu)] = \ln(h(\lambda,\mu)h(\mu,\lambda)). \tag{2.6}$$

The relation of these quantum fields to those ϕ_{A_2}, ϕ_{D_1} used in [1–4] are

$$\phi(\lambda) = \phi_{A_2}(\lambda) - \phi_{D_1}(\lambda) \qquad \Phi(\lambda) = \phi_{A_2}(\lambda) + \phi_{D_1}(\lambda). \tag{2.7}$$

The vacuum vector is normalized by unity $(0|0) = 1$.

The quantum fields (2.1) are linear combinations of the three canonical Bose fields. The derivative of the field $\psi(\lambda)$ will also be important:

$$\psi'(\lambda) \equiv \frac{\partial}{\partial\lambda}\psi(\lambda) = q'_\psi(\lambda) + p'_\psi(\lambda). \tag{2.8}$$

Non-zero commutation relations between the derivatives of annihilation operators $p(\lambda)$ and creation operators $q(\lambda)$ are:

$$[p'_\psi(\lambda), q_\phi(\mu)] = [p_\phi(\lambda), q'_\psi(\mu)] = \frac{2\mathrm{i}c}{c^2 + (\lambda-\mu)^2} \tag{2.9}$$

$$[p'_\psi(\lambda), q'_\psi(\mu)] = \left(\frac{1}{\lambda-\mu+\mathrm{i}c}\right)^2 + \left(\frac{1}{\mu-\lambda+\mathrm{i}c}\right)^2 \tag{2.10}$$

$$[p_\Phi(\lambda), q'_\psi(\mu)] = [p_\psi(\lambda), q'_\psi(\mu)] = -[p'_\psi(\lambda), q_\Phi(\mu)] = \frac{2(\mu-\lambda)}{(\lambda-\mu)^2 + c^2}. \tag{2.11}$$

It is worth mentioning that quantum fields (2.1) belong to the same Abelian sub-algebra. They all commute:

$$\begin{aligned}&[\psi(\lambda), \psi(\mu)] = [\psi(\lambda), \phi(\mu)] = [\psi(\lambda), \Phi(\mu)] = 0\\ &[\phi(\lambda), \Phi(\mu)] = [\phi(\lambda), \phi(\mu)] = [\Phi(\lambda), \Phi(\mu)] = 0.\end{aligned} \tag{2.12}$$

This property plays a very important role in the calculation of vacuum mean values in auxiliary Fock space.

In [1] the correlation function of local fields of the quantum nonlinear Schrödinger equation was represented as a mean value of a determinant of an integral operator, depending on the fields (2.1). At large space x and time t separation the determinant simplifies to [4]:

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_T = (0|Q(x,t)\left[1+\mathrm{o}\left(\frac{1}{\sqrt{t}}\right)\right]|0) \tag{2.13}$$

where $Q(x,t)$ is an operator in auxiliary Fock space

$$\begin{aligned} Q(x,t) = C([\phi(\lambda)],[\Phi(\lambda)])(2t)^{(\nu-1)^2/2}\mathrm{e}^{\psi(\lambda_0)-\mathrm{i}t\lambda_0^2-\mathrm{i}ht} \\ \times\exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,(|x-2\lambda t|-\mathrm{i}\,\mathrm{sign}(\lambda-\lambda_0)\psi'(\lambda))\right. \\ \left.\times\ln[1-\vartheta(\lambda)(1+\mathrm{e}^{\phi(\lambda)\,\mathrm{sign}(\lambda-\lambda_0)})]\right\}. \end{aligned} \tag{2.14}$$

Here C is a smooth bounded functional, which may depend on x and t only through the ratio $x/2t = \lambda_0$, which remains fixed. The Fermi weight $\vartheta(\lambda)$ is defined by

$$\vartheta(\lambda) = (1+\mathrm{e}^{\varepsilon(\lambda)/T})^{-1} \tag{2.15}$$

and

$$\nu = \frac{\mathrm{i}}{2\pi}\ln\{[1-\vartheta(\lambda_0)(1+\mathrm{e}^{-\phi(\lambda_0)})][1-\vartheta(\lambda_0)(1+\mathrm{e}^{\phi(\lambda_0)})]\}. \tag{2.16}$$

In this paper our main aim is to evaluate the mean value of the right-hand side of (2.14). Due to relations (2.5) and (2.6), the creation and annihilation parts of the fields ϕ and Φ commute with each other. Thus the only non-zero contribution to the vacuum mean value is provided by normal ordering of expressions containing the field $\psi(\lambda)$. It is easy to find the contribution of the factor $\mathrm{e}^{\psi(\lambda_0)}$ (see (2.14)). Indeed let us move this exponent to the left

$$(0|\mathrm{e}^{\psi(\lambda_0)} = (0|\mathrm{e}^{p_\psi(\lambda_0)}. \tag{2.17}$$

After this one can move $\mathrm{e}^{p_\psi(\lambda_0)}$ to the right using obvious relations:

$$[p_\psi(\lambda_0),\phi(\lambda_0)] = 0 \tag{2.18}$$

$$[p_\psi(\lambda_0),\nu] = 0 \tag{2.19}$$

$$\mathrm{e}^{p_\psi(\lambda_0)}\phi(\mu) = \left(\phi(\mu)+\ln\frac{h(\mu,\lambda_0)}{h(\lambda_0,\mu)}\right)\mathrm{e}^{p_\psi(\lambda_0)} \tag{2.20}$$

$$\mathrm{e}^{p_\psi(\lambda_0)}\Phi(\mu) = (\Phi(\mu)+\ln[h(\lambda_0,\mu)h(\mu,\lambda_0)])\mathrm{e}^{p_\psi(\lambda_0)} \tag{2.21}$$

$$\mathrm{e}^{p_\psi(\lambda_0)}\psi'(\mu) = \left(\psi'(\mu)+\frac{2(\mu-\lambda_0)}{(\mu-\lambda_0)^2+c^2}\right)\mathrm{e}^{p_\psi(\lambda_0)} \tag{2.22}$$

and

$$\mathrm{e}^{p_\psi(\lambda_0)}|0) = |0). \tag{2.23}$$

Thus we arrive at

$$\begin{aligned} (0|Q(x,t)|0) = (0|\tilde{C}([\phi(\lambda)],[\Phi(\lambda)])(2t)^{(\nu-1)^2/2}\mathrm{e}^{-\mathrm{i}t\lambda_0^2-\mathrm{i}ht} \\ \times\exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\left(|x-2\lambda t|-\mathrm{i}\,\mathrm{sign}(\lambda-\lambda_0)\left[\psi'(\lambda)+\frac{2(\lambda-\lambda_0)}{(\lambda-\lambda_0)^2+c^2}\right]\right)\right. \\ \left.\times\ln\left[1-\vartheta(\lambda)\left(1+\exp\left[\mathrm{sign}(\lambda-\lambda_0)\left(\phi(\lambda)+\ln\frac{h(\lambda,\lambda_0)}{h(\lambda_0,\lambda)}\right)\right]\right)\right]\right\}|0). \end{aligned} \tag{2.24}$$

Here the functional $\tilde{C}([\phi],[\Phi])$ can be obtained from the functional $C([\phi],[\Phi])$ by shifting of the arguments of the last one according to rules (2.20) and (2.21).

Now the right-hand side of (2.24) can be written in the following form

$$(0|Q(x,t)|0) = (0|\exp\left\{\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \psi'(\lambda) f(\lambda|\phi(\lambda))\right\} F([\phi],[\Phi])|0). \qquad (2.25)$$

Here

$$f(\lambda|\phi(\lambda)) = \frac{\operatorname{sign}(\lambda-\lambda_0)}{2\pi \mathrm{i}} \ln\left[1-\vartheta(\lambda)\left(1+\exp\left[\operatorname{sign}(\lambda-\lambda_0)\left(\phi(\lambda)+\ln\frac{h(\lambda,\lambda_0)}{h(\lambda_0,\lambda)}\right)\right]\right)\right] \qquad (2.26)$$

and

$$\begin{aligned} F([\phi],[\Phi]) = {} & \tilde{C}([\phi(\lambda)],[\Phi(\lambda)])(2t)^{(\nu-1)^2/2} \mathrm{e}^{-\mathrm{i}t\lambda_0^2-\mathrm{i}ht} \\ & \times \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \left(|x-2\lambda t| - 2\mathrm{i}\operatorname{sign}(\lambda-\lambda_0)\frac{\lambda-\lambda_0}{(\lambda-\lambda_0)^2+c^2}\right)\right. \\ & \left.\times \ln\left[1-\vartheta(\lambda)\left(1+\exp\left[\operatorname{sign}(\lambda-\lambda_0)\left(\phi(\lambda)+\ln\frac{h(\lambda,\lambda_0)}{h(\lambda_0,\lambda)}\right)\right]\right)\right]\right\}. \end{aligned} \qquad (2.27)$$

In the next section we shall evaluate the right-hand side of (2.25).

3. Evaluation of the mean value

The main purpose of this section is to evaluate the mean value

$$(0|\exp\left\{\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \psi'(\lambda) f(\lambda|\phi(\lambda))\right\} F([\phi],[\Phi])|0). \qquad (3.1)$$

Here complex function f becomes an operator, because it depends on quantum field $\phi(\lambda)$. It is worth mentioning that the particular case of (3.1), when $f(\lambda|\phi(\lambda))$ is a linear function of the field $\phi(\lambda)$, was first considered in [10].

We remind the reader of the definitions

$$\begin{aligned} \psi'(\lambda) &= p'_\psi(\lambda) + q'_\psi(\lambda) \\ \phi(\lambda) &= p_\phi(\lambda) + q_\phi(\lambda) \\ \Phi(\lambda) &= p_\Phi(\lambda) + q_\Phi(\lambda). \end{aligned} \qquad (3.2)$$

As usual, the relations $p(\lambda)|0) = 0$ and $(0|q(\lambda) = 0$ for all p and q are satisfied as well as the commutation relations

$$[p'_\psi(\lambda), q_\phi(\mu)] = \xi(\lambda,\mu) = [p_\phi(\lambda), q'_\psi(\mu)] \qquad (3.3)$$

$$[p'_\psi(\lambda), q_\Phi(\mu)] = \tilde{\xi}(\lambda,\mu) = -[p_\Phi(\lambda), q'_\psi(\mu)] \qquad (3.4)$$

$$[p'_\psi(\lambda), q'_\psi(\mu)] = \eta(\lambda,\mu). \qquad (3.5)$$

The complex functions $\xi(\lambda,\mu)$ and $\tilde{\xi}(\lambda,\mu)$ are equal to

$$\xi(\lambda,\mu) = \xi(\mu,\lambda) = \frac{2\mathrm{i}c}{c^2+(\lambda-\mu)^2} \qquad \tilde{\xi}(\lambda,\mu) = \frac{2(\lambda-\mu)}{(\lambda-\mu)^2+c^2} \qquad (3.6)$$

and

$$\eta(\lambda,\mu) = \left(\frac{1}{\lambda-\mu+\mathrm{i}c}\right)^2 + \left(\frac{1}{\lambda-\mu-\mathrm{i}c}\right)^2. \qquad (3.7)$$

However, we do not use explicit expressions (3.6) and (3.7) in this section.

We evaluate the mean value (3.1) in three steps. First, we consider an auxiliary problem—the scalar case.

3.1. Scalar case

Consider the following mean value

$$(0|\mathrm{e}^{\{\alpha\psi'(\lambda)f(\phi(\mu))\}}F([\phi],[\Phi])|0). \tag{3.8}$$

Here $F([\phi],[\Phi])$ is a smooth functional, depending on the fields ϕ and Φ. A complex function $f(z)$ becomes an operator-valued function because its argument $\phi(\mu)$ is an operator. As usual such an expression should be understood as a formal Taylor series; therefore, we assume that $f(z)$ is homomorphic within some circle $|z|<\rho_0$.

The parameter α is a complex number such that the following restriction holds:

$$|\alpha|\cdot|f(z\xi(\lambda,\mu))|<|z| \qquad \text{for } |z|=\rho<\frac{\rho_0}{|\xi(\lambda,\mu)|}. \tag{3.9}$$

The complex numbers λ and μ are fixed.

Let us decompose the exponent in a Taylor series,

$$\mathrm{e}^{\{\alpha\psi'(\lambda)f(\phi(\mu))\}}=\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}(\psi'(\lambda))^n f^n(\phi(\mu)). \tag{3.10}$$

We would like to emphasize that here we essentially used the commutativity of the fields ψ' and ϕ.

In order to calculate $(0|(\psi'(\lambda))^n$ we use the Cauchy integral representation

$$(\psi'(\lambda))^n=\frac{\mathrm{d}^n}{\mathrm{d}z^n}\mathrm{e}^{z\psi'(\lambda)}\bigg|_{z=0}=\frac{n!}{2\pi\mathrm{i}}\int_{|z|=\rho}\mathrm{d}z\,\frac{\mathrm{e}^{z\psi'(\lambda)}}{z^{n+1}}. \tag{3.11}$$

The evaluation of $(0|\mathrm{e}^{z\psi'(\lambda)}$ is a standard problem in quantum field theory

$$(0|\mathrm{e}^{z\psi'(\lambda)}=\mathrm{e}^{z^2\eta(\lambda,\lambda)/2}(0|\mathrm{e}^{zp'_\psi(\lambda)}. \tag{3.12}$$

After substituting this into (3.11), we obtain

$$(0|(\psi'(\lambda))^n=\frac{n!}{2\pi\mathrm{i}}\int_{|z|=\rho}\mathrm{d}z\,\frac{\mathrm{e}^{z^2\eta(\lambda,\lambda)/2}}{z^{n+1}}(0|\mathrm{e}^{zp'_\psi(\lambda)}. \tag{3.13}$$

Further substitution into (3.8) and (3.10) gives the expression for the mean value

$$(0|\mathrm{e}^{\{\alpha\psi'(\lambda)f(\phi(\mu))\}}F([\phi],[\Phi])|0)$$
$$=\sum_{n=0}^{\infty}\frac{\alpha^n}{2\pi\mathrm{i}}\int_{|z|=\rho}\mathrm{d}z\,\frac{\mathrm{e}^{z^2\eta(\lambda,\lambda)/2}}{z^{n+1}}(0|\mathrm{e}^{zp'_\psi(\lambda)}f^n(\phi(\mu))F([\phi],[\Phi])|0). \tag{3.14}$$

Recall that creation and annihilation parts of the fields ϕ and Φ commute with each other; therefore, we have

$$(0|\mathrm{e}^{zp'_\psi(\lambda)}f^n(\phi(\mu))F([\phi],[\Phi])|0)=(0|\mathrm{e}^{zp'_\psi(\lambda)}f^n(q_\phi(\mu))F([q_\phi],[q_\Phi])|0)$$
$$=f^n(z\xi(\lambda,\mu))(0|\mathrm{e}^{zp'_\psi(\lambda)}F([\phi],[\Phi])|0). \tag{3.15}$$

If we substitute this into (3.14) we may calculate the sum with respect to n. Due to restriction (3.9) this series is absolutely convergent:

$$(0|\mathrm{e}^{\alpha\psi'(\lambda)f(\phi(\mu))}F([\phi],[\Phi])|0)=\frac{1}{2\pi\mathrm{i}}\int_{|z|=\rho}\mathrm{d}z\,\frac{\mathrm{e}^{z^2\eta(\lambda,\lambda)/2}}{z-\alpha f(z\xi(\lambda,\mu))}(0|\mathrm{e}^{zp'_\psi(\lambda)}F([\phi],[\Phi])|0). \tag{3.16}$$

Due to relation (3.9) and the Rouché theorem, the equation

$$z - \alpha f(z\xi(\lambda,\mu)) = 0 \tag{3.17}$$

has exactly one zero of the first order $z = z_0 = z_0(\alpha)$ in the domain $|z| < \rho$. Let us emphasize that (3.17) is a classical equation where only complex functions are involved (i.e. no quantum operators). Thus, taking the integral with respect to z, we arrive at

$$(0|\mathrm{e}^{\{\alpha\psi'(\lambda)f(\phi(\mu))\}}F([\phi],[\Phi])|0) = \frac{\mathrm{e}^{z_0^2\eta(\lambda,\lambda)/2}(0|\mathrm{e}^{z_0 p'_\psi(\lambda)}F([\phi],[\Phi])|0)}{1-\frac{\partial}{\partial z_0}f(z_0\xi(\lambda,\mu))}. \tag{3.18}$$

The right-hand side may be further simplified

$$\mathrm{e}^{z_0^2\eta(\lambda,\lambda)/2}(0|\mathrm{e}^{z_0 p'_\psi(\lambda)}F([\phi],[\Phi])|0) = (0|\mathrm{e}^{z_0\psi'(\lambda)}F([\phi],[\Phi])|0). \tag{3.19}$$

Let us again recall that in the original formula (3.8) $f(\phi(\mu))$ was a quantum operator since it depends on the quantum field $\phi(\mu)$. The result of the above calculation shows that this function may be replaced by a complex number z_0.

It is worth mentioning that one can analytically continue the result obtained with respect to α into the domain where the inequality (3.9) is not valid. We propose the following theorem.

Theorem 3.1.

$$(0|\mathrm{e}^{\{\alpha\psi'(\lambda)f(\phi(\mu))\}}F([\phi],[\Phi])|0) = \frac{(0|\mathrm{e}^{z_0\psi'(\lambda)}F([\phi],[\Phi])|0)}{1-\alpha f'(z_0\xi(\lambda,\mu))}. \tag{3.20}$$

Here the complex number z_0 can be found from the equation

$$z_0 = \alpha f(z_0\xi(\lambda,\mu)) \tag{3.21}$$

and

$$f'(z_0\xi(\lambda,\mu)) = \frac{\partial}{\partial z} f(z\xi(\lambda,\mu))\bigg|_{z=z_0}. \tag{3.22}$$

Remark. Equation (3.21) may have many solutions if we do not impose restriction (3.9). In this case one should choose the solution $z_0 = z_0(\alpha)$, with the property:

$$z_0(\alpha)|_{\alpha=0} = 0. \tag{3.23}$$

Let us remind the reader that no operators are involved in equation (3.21) since $f(z)$ is a complex function. This is the complex equation for the complex number z_0.

3.2. Matrix case

The method of calculating the mean value described above can be easily generalized for more complicated cases. Namely let us consider the example:

$$(0|\exp\left\{\sum_{k=1}^{N}\psi'(\lambda_k)f_k(\phi(\lambda_k))\right\}F([\phi],[\Phi])|0). \tag{3.24}$$

It is clear that we may find the mean value in an analogous fashion to that for a scalar case. Let us briefly describe the main steps of the corresponding derivation.

First, we have

$$\exp\left\{\sum_{k=1}^{N}\psi'(\lambda_k)f_k(\phi(\lambda_k))\right\} = \prod_{j=1}^{N}\sum_{n_j=0}^{\infty}\frac{1}{n_j!}(\psi'(\lambda_j))^{n_j}f_j^{n_j}(\phi(\lambda_j)). \tag{3.25}$$

The normal ordering of $\psi'(\lambda_j)$ may be performed in a similar fashion to (3.11) and (3.12):

$$(0|\prod_{j=1}^{N}\left[\sum_{n_j=0}^{\infty}\frac{1}{n_j!}(\psi'(\lambda_j))^{n_j}\right]=\frac{1}{(2\pi\mathrm{i})^N}\int\prod_{j=1}^{N}\frac{\mathrm{d}z_j}{z_j^{n_j+1}}$$
$$\times\exp\left\{\frac{1}{2}\sum_{j,k=1}^{N}z_jz_k\eta(\lambda_j,\lambda_k)\right\}(0|\mathrm{e}^{\sum_{k=1}^{N}z_kp'_\psi(\lambda_k)}. \tag{3.26}$$

Here each of the integrals is taken by circle, which appears to be a common domain of analyticity of the functions $f_k(z)$.

Next we find the mean value

$$(0|\mathrm{e}^{\sum_{k=1}^{N}z_kp'_\psi(\lambda_k)}\prod_{k=1}^{N}f_k^{n_k}(\phi(\lambda_k))F([\phi],[\Phi])|0)$$
$$=\prod_{k=1}^{N}f_k^{n_k}\left(\sum_{j=1}^{N}z_j\xi(\lambda_j,\lambda_k)\right)(0|\mathrm{e}^{\sum_{k=1}^{N}z_kp'_\psi(\lambda_k)}F([\phi],[\Phi])|0). \tag{3.27}$$

Now we substitute this into the expression for our mean value (3.24) and sum with respect to each n_k:

$$(0|\mathrm{e}^{\sum_{k=1}^{N}\psi'(\lambda_k)f_k(\phi(\lambda_k))}F([\phi],[\Phi])|0)$$
$$=\frac{1}{(2\pi\mathrm{i})^N}\int\prod_{j=1}^{N}\mathrm{d}z_j\,\frac{\mathrm{e}^{1/2\sum_{j,k=1}^{N}z_jz_k\eta(\lambda_j,\lambda_k)}(0|\mathrm{e}^{\sum_{k=1}^{N}z_kp'_\psi(\lambda_k)}F([\phi],[\Phi])|0)}{\prod_{j=1}^{N}[z_j-f_j(\sum_{m=1}^{N}z_m\xi(\lambda_m,\lambda_j))]}. \tag{3.28}$$

In order to take the z_j integral we introduce the complex numbers z_j^0 as solutions of the system

$$z_j^0=f_j\left(\sum_{m=1}^{N}z_m^0\xi(\lambda_m,\lambda_j)\right). \tag{3.29}$$

We also define the matrix M:

$$M_{jk}=\delta_{jk}-\frac{\partial}{\partial z_j}f_k\left(\sum_{m=1}^{N}z_m\xi(\lambda_m,\lambda_k)\right)\bigg|_{z_l=z_l^0}. \tag{3.30}$$

After evaluating the z_j integration, we obtain the result

$$(0|\exp\left\{\sum_{k=1}^{N}\psi'(\lambda_k)f_k(\phi(\lambda_k))\right\}F([\phi],[\Phi])|0)=\frac{(0|\exp\{\sum_{k=1}^{N}\psi'(\lambda_k)z_k^0\}F([\phi],[\Phi])|0)}{\det M}. \tag{3.31}$$

As in the scalar case, system (3.29) may have many solutions. In this case one can consider the replacement $f_k\to\alpha f_k$. After this, it is necessary to choose the solution of (3.29), which approaches zero as $\alpha\to 0$ and to continue this solution to the point $\alpha=1$.

3.3. Continuous case

In order to evaluate the mean value (3.1) let us consider the continuous limit of (3.31):

$$\lambda_{k+1}=\lambda_k+\Delta \tag{3.32}$$
$$f_k(\phi(\lambda_k))=\Delta f(\lambda_k|\phi(\lambda_k)) \tag{3.33}$$
$$z_k^0=\Delta z(\lambda_k) \tag{3.34}$$

and take the limit $\Delta \to 0$. We notice that the system of constraints in (3.29) turns into the integral equation:

$$z(\lambda) = f\left(\lambda \Big| \int_{-\infty}^{\infty} \mathrm{d}\mu\, z(\mu)\xi(\mu,\lambda)\right). \tag{3.35}$$

Furthermore, the matrix M_{jk} becomes an integral operator with the kernel

$$M(\lambda,\mu) = \delta(\lambda-\mu) - \frac{\delta}{\delta z(\mu)} f\left(\lambda \Big| \int_{-\infty}^{\infty} \mathrm{d}s\, z(s)\xi(s,\lambda)\right). \tag{3.36}$$

Equation (3.31) has the following continuous limit:

$$\begin{aligned}(0|\exp&\left\{\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\psi'(\lambda)f(\lambda|(\phi(\mu))\right\}F([\phi],[\Phi])|0)\\ &= (\det M)^{-1}(0|\exp\left\{\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\psi'(\lambda)z(\lambda)\right\}F([\phi],[\Phi])|0).\end{aligned} \tag{3.37}$$

Equations (3.35)–(3.37) are the main result of this section. We would like to emphasize that the commutativity of the quantum fields (2.1) is extremely important for obtaining of this result.

The following calculations are trivial. We have

$$\begin{aligned}(0|\exp\left\{\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\psi'(\lambda)z(\lambda)\right\} &= \exp\left\{\tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\mathrm{d}\mu\,\eta(\lambda,\mu)z(\lambda)z(\mu)\right\}\\ &\quad\times(0|\exp\left\{\int_{-\infty}^{\infty}\mathrm{d}\lambda\,p'_\psi(\lambda)z(\lambda)\right\}.\end{aligned} \tag{3.38}$$

The action of the operator $\exp\{p'_\psi\}$ on the functional F leads to the shift of the arguments of the last one (see (2.20), (2.21)). We find

$$\begin{aligned}(0|\exp&\left\{\int_{-\infty}^{\infty}\mathrm{d}\lambda\,p'_\psi(\lambda)z(\lambda)\right\}F([\phi],[\Phi])|0)\\ &= F\left(\left[\int_{-\infty}^{\infty}\mathrm{d}\lambda\,z(\lambda)\xi(\lambda,\mu)\right],\left[\int_{-\infty}^{\infty}\mathrm{d}\lambda\,z(\lambda)\tilde{\xi}(\lambda,\mu)\right]\right).\end{aligned} \tag{3.39}$$

In the next section we shall use these results in order to evaluate the mean value of the asymptotic expression (2.24).

4. The mean value of the leading term

In order to find the mean value of the leading term of asymptotics (2.24), we need only to substitute the concrete expressions (2.26), (2.27) into equations (3.35)–(3.37) and (3.39).

An integral equation for the z-function is

$$z(\lambda) = -\frac{\mathrm{i}}{2\pi}\,\mathrm{sign}(\lambda-\lambda_0)\ln\{1-\vartheta(\lambda)X(\lambda,\lambda_0)\} \tag{4.1}$$

where

$$X(\lambda,\lambda_0) \equiv 1 + \exp\left\{\mathrm{sign}(\lambda-\lambda_0)\left(\ln\frac{h(\lambda,\lambda_0)}{h(\lambda_0,\lambda)} + \int_{-\infty}^{\infty}\mathrm{d}\mu\,\frac{2\mathrm{i}cz(\mu)}{c^2+(\lambda-\mu)^2}\right)\right\}. \tag{4.2}$$

The integral operator M has the kernel

$$M(\lambda,\mu) = \delta(\lambda-\mu) - \frac{\mathrm{i}}{2\pi}\,\mathrm{sign}(\lambda-\lambda_0)\frac{\delta}{\delta z(\mu)}\ln\{1-\vartheta(\lambda)X(\lambda,\lambda_0)\}. \tag{4.3}$$

After evaluating the variation derivative one should set $z(\lambda)$ equal to the solution of (4.1).

The functional $F([\phi],[\Phi])$ is given in (2.27). It consists of three factors: an exponential factor, a power law correction with respect to t, and a constant factor, which depends only on the ratio $x/2t=\lambda_0$. Using (3.39) and the integral equation (4.1), we find the exponential factor to be

$$\exp\left\{-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,(x-2\lambda t)z(\lambda)\right\}. \tag{4.4}$$

The next factor is a power law correction

$$(2t)^{(\nu-1)^2/2} \tag{4.5}$$

where the expression for ν follows from (2.16):

$$\nu=\frac{\mathrm{i}}{2\pi}\ln\{[1-\vartheta(\lambda_0)(1+\mathrm{e}^{-\phi(\lambda_0)})][1-\vartheta(\lambda_0)(1+\mathrm{e}^{\phi(\lambda_0)})]\}. \tag{4.6}$$

Instead of $\phi(\lambda_0)$ we substitute the integral expression,

$$\phi(\lambda_0)\to u(\lambda_0)=\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\frac{2\mathrm{i}c}{c^2+(\lambda_0-\lambda)^2}z(\lambda) \tag{4.7}$$

where the function $z(\lambda)$ is a solution of (4.1). So the power law correction becomes

$$(2t)^{(\tilde\nu-1)^2/2} \tag{4.8}$$

where

$$\tilde\nu=\frac{\mathrm{i}}{2\pi}\ln\{[1-\vartheta(\lambda_0)(1+\mathrm{e}^{-u(\lambda_0)})][1-\vartheta(\lambda_0)(1+\mathrm{e}^{u(\lambda_0)})]\}. \tag{4.9}$$

Finally, the constant term is equal to

$$g=\tilde C([u(\mu)],[v(\mu)])\mathrm{e}^{v(\lambda_0)}\exp\left\{\tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\mathrm{d}\mu\,\eta(\lambda,\mu)z(\lambda)z(\mu)\right\} \tag{4.10}$$

where

$$v(\mu)=\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\frac{2(\lambda-\mu)}{c^2+(\lambda_0-\lambda)^2}z(\lambda). \tag{4.11}$$

Thus we obtain for the mean value (2.24) of the operator $Q(x,t)$

$$(0|Q(x,t)|0)=g\cdot\mathrm{e}^{-\mathrm{i}t\lambda_0^2-\mathrm{i}ht}(2t)^{(\tilde\nu-1)^2/2}(\det M)^{-1}\exp\left\{-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,(x-2\lambda t)z(\lambda)\right\}. \tag{4.12}$$

This is the main result of this paper.

Expression (4.12) is the leading term of the asymptotic evaluation of the Fredholm determinant representing the correlation function [1]. In the next publication we shall study the corrections. The mean value of corrections can contribute to the leading term of the asymptotic behaviour of the correlation function.

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